Affine Periodic Solutions in Distribution of Stochastic Differential Equations

Xiaomeng Jiang

Joint work with Y. Li & X. Yang

Changchun, Jilin

2019.7

Outline

Question: do periodic systems admit solutions with the same period?

Origin

- Kepler's Law & Two Body Problem
  - J. Kepler. *Astronomia Nova*, 1609.
  - I. Newton. *Philosophiae Naturalis Principia Mechanics*, 1687.
- Periodic Systems & Lyapunov's Method
  - H. Poincaré. *Les Méthodes Nouvelles de la Mécanique Céleste, Vol. I, II, III*, 1890s.
  - A. M. Lyapunov. *The General Problem of the Stability of Motion* (in Russian), 1892.

Deterministic Periodic Systems

Consider ordinary differential equation (ODE)

$$x'(t) = f(t, x(t)) \tag{1}$$

in $\mathbb{R}^l$, $l < \infty$. Function f satisfies

$$f(t+T, x) = f(t, x), \ T > 0 \quad \forall \, t \in \mathbb{R}, \ x \in \mathbb{R}^l. \tag{2}$$

Question: does boundedness of the solution ensures the existence of periodic solutions?

Boundedness $\Rightarrow$ Periodicity?

- $l = 2$: Boundedness of solutions $\Rightarrow$Existence of periodic solutions.

  Massera's criterion.

  - J. L. Massera. Duke Math. J., 1950.

- $l > 2$: Boundedness **fails** the sufficiency.

  Halanay's criterion.

  $$\lim_{t \to +\infty} [x(t+T) - x(t)] = 0.$$

  - A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags*, 1966.

Boundedness $\Rightarrow$ Periodicity?

LaSalle stationary oscillation priciple.

There exists a continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+ \backslash \{0\}$ satisfying

$$\lim_{t \to \infty} a(t) = 0,$$

such that

$$|x(t) - y(t)| \leq a(t)|x_0 - y_0| \quad \forall t \geq 0.$$

- J. LaSalle, S. Lefschetz. *Stability by Lyapunov's Direct Method*, 1961.

Outline

Stochastic Periodic Systems

- Stochastic differential equation (SDE).

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t). \tag{3}$$

- Fokker-Planck equation (FPE).

$$\begin{aligned} \frac{\partial}{\partial t} p(t,x) = &- \sum_i \frac{\partial}{\partial x_i} \big(f_i(t,x) p(t,x) \big) \\ &+ \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \big((gg^\top)_{ij}(t,x) p(t,x) \big). \end{aligned} \tag{4}$$

- Effect of noise: mixing and averaging.
- General hypotheses:

(H1) f and g satisfy linear growth condition.
(H2) f and g satisfy global Lipschitz condition.

Question: does SDE admit periodic solutions under stochastic boundedness?

- **Recurrence**
  - A. Kolmogorov. Math. Ann., 1936.
- **Stationary measure**
  - Existence
    - L. Hörmander. Acta Math., 1967.
    - W. Huang, M. Ji, Z. Liu, Y. Yi. Ann. Probab., 2015.
    - W. Huang, M. Ji, Z. Liu, Y. Yi. JDDE., 2015-2016.
  - Regularity
    - V. I. Bogachev, N. V. Krylov, M. Röchner. CPDE., 2001.

Stochastic Periodic Systems

Problems:

- Random motion of white noise $\longrightarrow$ periodicity in distribution.
  - Periodicity in Distribution:

$$X(t+T) \xlongequal{d} X(t) \ \forall\ t \in \mathbb{R}^+. \tag{5}$$

- Non compactness of space of probability measures $\longrightarrow$ effective tools.
- Related works
  - R. Khasminskii. *Stochastic Stability of Differential Equations*, 2012.
  - C. Feng, H. Zhao. J. Funct. Anal., 2012.

- **Recent works**
  - Z. Liu, W. Wang. JDE., 2016.
    Favard separation method for almost periodic SDEs.
  - F. Chen, Y. Han, Y. Li, X. Yang. JDE., 2017.
    Weak Halanay's criterion for periodic SDEs.

A Viewpoint in Probability Measure Space

State Space		**Probability Measure Space**
SDE	$\longrightarrow$	FPE
Stationary Measures	$\longrightarrow$	Constant Point
Periodic Solutions	$\longrightarrow$	Closed Orbit
Quasi Periodic Solutions	$\longrightarrow$	Torus

Outline

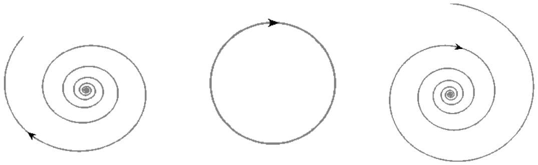
Affine Periodicity

Assume that f in ODE (1) is (Q,T)-affine periodic. Namely,

$$f(t+T,x) = Qf(t,Q^{-1}x) \tag{6}$$

for some $Q \in GL(l)$, $T > 0$ and all $t \in \mathbb{R}$, $x \in \mathbb{R}^l$.

- (Q,T)-affine periodic solutions: $x(t+T) = Qx(t) \quad \forall t \in \mathbb{R}^+$.
- $Q = I_l$: periodic system.
- $Q = -I_l$: anti-periodic system.
- $Q \in O(l)$: rotation of a rigid body.
- Example: $x(t) = e^{at}(\cos \omega t, \sin \omega t)^\top$.



(a) $a > 0, \omega < 0$ (b) $a = 0, \omega < 0$ (c) $a < 0, \omega < 0$

Works on affine periodic ordinary differential equations.

- Anti-periodicity
  - R. Wu, F. Cong, Y. Li. Appl. Math. Lett., 2011.
  - T. Haddad, T. Haddad. Electron. J. Differential Equations, 2013.
  - O. Chadli, Q. H. Ansari, J. C. Yao. J. Optim. Theory Appl., 2016.
- Affine perodicity
  - Y. Li, F. Huang. Adv. Nonlinear Stud., 2015.
  - X. Chang, Y. Li. DCDS., 2016.
  - H. Wang, X. Yang, Y. Li, X. Li. DCDS Ser. B, 2017.
  - J. Xing, X. Yang, Y. Li. SCIENCE CHINA Mathematics, 2018.

Outline

Question: does SDE admit affine periodic solutions in distribution?

Consider SDE (3)

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t),$$

with f and g are continuous satisfying

$$f(t+T, x) = Qf(t, Q^{-1}x),\ g(t+T, x) = Qg(t, Q^{-1}x),\ Q \in O(l),\ T > 0.$$

- (Q, T)-affine periodic solutions in distribution:

$$X(t+T) \xlongequal{d} QX(t)\ \forall\ t \in \mathbb{R}^+.$$

Stochastic Affine Periodic Systems

Theorem 1 (Weak stochastic Halanay's criterion)

Assume that f and g satisfy **(H1)-(H2)**. Moreover, assume that

(H3) (Q,T)-affine periodic boundedness:

$$\mathbb{E}|Q^{-n}X(nT)|^2 \leq C \text{ for some } C > 0 \text{ and all } n \geq 0.$$

(H4) weak stochastic affine periodic Halanay's criterion:

$$\lim_{k\to\infty} \frac{1}{n_k+1} \sum_{N=0}^{n_k} d_{BL}\left(p_{Q^{-N-1}X(NT+T)}, p_{Q^{-N}X(NT)}\right) = 0,$$

where $p_{X(t)}$ is the probability density of X at t and d_{BL} is a probability measure distance in the sense of duality.

Then (3) admits a (Q,T)-affine periodic solution in distribution.

(H4)' $\lim_{k\to\infty} \frac{1}{n_k+1} \sum_{N=0}^{n_k} \mathbb{E}\left|Q^{-N-1}X(NT+T) - Q^{-N}X(NT)\right|^2 = 0.$

Stochastic Affine Periodic Systems

Here,

$$
\begin{aligned}
d_{BL}(\mu, \gamma) &= \sup_{\|h\|_{BL} \le 1} \left| \int h d(\mu - \gamma) \right|, \\
\|h\|_{BL} &= \max\{\|h\|_\infty, \|h\|_L\}, \\
\|h\|_\infty &= \sup_{x \in \mathbb{R}^l} |h(x)|, \\
\|h\|_L &= \sup_{x, y \in \mathbb{R}^l, x \neq y} \frac{|h(x) - h(y)|}{|x - y|}.
\end{aligned}
$$

Proof of Theorem 1

Step 1. Construct of possible initial state.

$$y_k = Q^{-\xi_k} X_\eta(\xi_k T).$$

- X_η: solution of (3) starting from η.
- ξ_k: independent of W and η satisfying

$$\mathbb{P}\{\xi_k = N\} = \frac{1}{k+1},\ N = 0, \ldots, k,\ k \in \mathbb{N}_+.$$

$\Rightarrow$

- $\mathbb{P}\{y_k > M\} \to 0$ as $M \to \infty$.
- $\exists(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $\{\tilde{y}_{n_k}\}$ and $\tilde{y}_0$ s.t. $\boldsymbol{y_{n_k} \overset{d}{=\!=} \tilde{y}_{n_k}}$ and $\boldsymbol{\tilde{y}_{n_k} \xrightarrow{d} \tilde{y}_0}$.
- By **(H3)**, $\boldsymbol{\mathbb{E}|\tilde{y}_{n_k}|^2 = \mathbb{E}|y_{n_k}|^2 \leq C}$.

Take $Y_{n_k}(t) = Q^{-\xi_{n_k}} X_\eta(t + \xi_{n_k} T)$.

Step 2. A subsequence of Y_{n_k} converges in distribution on $[0, T]$.

- When $\xi_k = N$,

$$\begin{cases} dY_k(t) = f(t, Y_k(t))dt + g(t, Y_k(t))dW_N(t), \\ Y_k(0) = y_k. \end{cases}$$

  - $W_N(t) = W(t + NT) - W(NT)$, $W_{\xi_k} \stackrel{d}{=\!=} W$.
- $\exists r_0$ s.t. $\|Y_k\|_{2,[0,T]} \leq r_0$.
- Theorem 3.1, Liu & Wang, JDE, 2016 $\Rightarrow \exists \{Y_{n_k}\}$ and a weak solution (Z, W) of (3) s.t. $Y_{n_k} \xrightarrow{d} Z$ and $\|Z\|_{2,[0,T]} \leq r_0$.

Theorem 3.1. Liu & Wang, JDE., 2016.

Consider SDEs

$$\begin{cases} dX_n(t) = f_n(t, X_n(t))dt + g_n(t, X_n(t))dW(t), \\ X_n(0) = \xi_n, \end{cases}$$

satisfying **(H1)-(H2)** and

(i). $f_n \to f$ and $g_n \to g$ point-wise on $\mathbb{R}^+ \times \mathbb{R}^n$ as $n \to \infty$,

(ii). there exist a common $r_0 > 0$ such that $\sup_{t \in \mathbb{R}^+} \mathbb{E}|X_n(t)|^2 \leq r_0$.

Then there exists a subsequence $\{X_{n_k}\}$, a stochastic process X and a Brownian motion B such that $X_{n_k} \xrightarrow{d} X$ uniformly on any compact intervals and (X, B) is a solution of SDE (3) with $\sup_{t \in \mathbb{R}^+} \mathbb{E}|X(t)|^2 \leq r_0$.

Proof of Theorem 1

Step 3. Show that Z is (Q,T)-affine periodic in distribution.

- **(H4)** $\Rightarrow$ $\boldsymbol{Q^{-1}Z(T) \overset{d}{=\!=} Z(0)}$.
- $(Q^{-1}Z(\cdot+T), W_1)$ is also a weak solution of (3).
- Uniqueness of weak solution $\Rightarrow$ Z is (Q,T)-affine periodic.

Theorem 2 (Stochastic LaSalle type stationary oscillation principle)

Assume that f and g satisfy **(H1)-(H2)**. Moreover, assume that

(H5) there is a continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+ \backslash \{0\}$ with $r := \overline{\lim_{k\to\infty}} a(kT) < 1$ such that for any solutions X and Y of (3),

$$d_{BL}\left(p_{X(t)}, p_{Y(t)}\right) \leq a(t) d_{BL}\left(p_{X(0)}, p_{Y(0)}\right),$$

Then (3) admits a unique asymptotically stable (Q, T)-affine periodic solution in distribution.

(H5)' There is a continuous function $a : \mathbb{R}^+ \to \mathbb{R}\backslash\{0\}$ with $\lim_{k\to\infty} a(kT) = 0$ such that

$$\mathbb{E}|X(t) - Y(t)|^2 \leq a(t)\mathbb{E}|X(0) - Y(0)|^2. \qquad (7)$$

Proof of Theorem 2

Define Poincaré mapping P as

$$P(p_\xi) = p_{Q^{-1}X_\xi(T)}.$$

- $P^k(p_\xi) = p_{Q^{-k}X_\xi(kT)}$.
- P is an r_1-contraction mapping of order k_0, i.e., $\exists\ r_1 \in (r, 1)$ and $k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0$

$$d_{BL}\left(P^k(p_\xi), P^k(p_\lambda)\right) \leq r_1 d_{BL}(p_\xi, p_\lambda).$$

- Contraction mapping fixed point theorem $\Rightarrow \exists\ \xi_0$ unique in distribution such that $\xi_0 \stackrel{d}{=\!=} Q^{-1}X_{\xi_0}(T)$. X_{ξ_0} is (Q, T)-affine periodic in distribution.
- Contraction $\Rightarrow$ asymptotic stability.

Stochastic Affine Periodic Systems

Theorem 3 (Existence via Lyapunov's method)

Suppose f and g satisfy **(H1)-(H2)**. Moreover, assume that there exists a Lyapunov function $V : \mathbb{R}^+ \times \mathbb{R}^l \to \mathbb{R}$ satisfying following conditions:

(H6) there are $A, B > 0$ constant such that,

$$A|x-y|^2 \leq V(t, x-y) \leq B|x-y|^2 \quad \forall\, x,\ y \in \mathbb{R}^l.$$

(H7) There exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ locally integrable satisfying $\int_0^\infty \alpha(s)ds \to -\infty$ such that

$$\mathcal{L}V(t, x-y) \leq \alpha(t)V(t, x-y) \quad \forall (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^l \times \mathbb{R}^l.$$

Then (3) admits a unique mean square asymptotically stable (Q, T)-affine periodic solution in distribution.

Stochastic Affine Periodic Systems

Here, operator $\mathcal{L}$ is defined as

$$\begin{aligned}
&\mathcal{L}V(t,x-y) \\
&:=\frac{\partial V}{\partial t}(t,x-y)+\left\langle f(t,x)-f(t,y),\frac{\partial V}{\partial x}(t,x-y)\right\rangle \\
&\quad+\frac{1}{2}\,\mathbf{tr}\left[(g(t,x)-g(t,y))^{\top}\,\mathbf{Hess}\,V(t,x-y)\,(g(t,x)-g(t,y))\right].
\end{aligned}$$

Theorem 4 (Existence via Lyapunov's method & LaSalle's principle)

Assume that f and g satisfy **(H1)-(H2)**. Moreover, assume

(H8) there are C^1 real positive definite matrix-valued function $D : \mathbb{R}^+ \to \mathbb{R}^{l\times l}$ with $D = D^\top$ and locally integrable bounded from above function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ with $\int_0^{kT} \alpha(s)ds \to -\infty$ as $k \to \infty$, such that for any $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}^l$,

$$D'(t) + 2\,\mathbf{J}_f^\top(t,x)D(t) + \sum_{i=1}^{m} \mathbf{J}_{g_i}^\top(t,x)D(t)\,\mathbf{J}_{g_i}(t,y) < \alpha(t)D(t),$$

$$D(kT) = D(0) \;\; \forall \;\; k \in \mathbb{N},$$

where $\mathbf{J}$ is the Jacobian matrix with respect to x.

Then (3) admits a unique asymptotically stable (Q,T)-affine periodic solution in distribution.

Thank you very much!